Learning Objectives/Syllabus Covered:

- 1.3.1 Use implication, converse, equivalence, negation, inverse, contrapositive

Cambridge 6B

The negation of a statement

To **negate** a statement P we write its very opposite, which we call '**not** P'. For example, consider the following four statements and their negations.

P		not P	
The sky is green.	(false)	The sky is not green.	(true)
$1+1=2$	(true)	$1+1\neq 2$	(false)
All prime numbers are odd.	(false)	There exists an even prime number.	(true)
All triangles have three sides.	(true)	Some triangle does not have three sides.	(false)

Notice that negation turns a true statement into a false statement, and a false statement into a true statement.

Example 6

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $2>1$

b 5 is divisible by 3

c The sum of any two odd numbers is even.

d There are two primes whose product is 12.

Example 6

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $2 > 1$

b 5 is divisible by 3

c The sum of any two odd numbers is even.

d There are two primes whose product is 12.

Solution

a P: $2 > 1$ (true)

not P: $2 \leq 1$ (false)

b P: 5 is divisible by 3 (false)

not P: 5 is not divisible by 3 (true)

c P: The sum of any two odd numbers is even. (true)

not P: There are two odd numbers whose sum is odd. (false)

d P: There are two primes whose product is 12. (false)

not P: There are no two primes whose product is 12. (true)

Augustus De Morgan (27 June 1806 – 18 March 1871) was a British mathematician and logician.

He formulated De Morgan's laws and introduced the term mathematical induction, making its idea rigorous.

Use sets to "discover" his laws

De Morgan's laws

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

De Morgan's laws

not (P *and* Q) is the same as (not P) *or* (not Q)

not (P *or* Q) is the same as (not P) *and* (not Q)

Example 7

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 6 is divisible by 2 and 3

b 10 is divisible by 2 or 7

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Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 6 is divisible by 2 and 3

b 10 is divisible by 2 or 7

Solution

a	P:	6 is divisible by 2 and 6 is divisible by 3	(true)
	not P:	6 is not divisible by 2 or 6 is not divisible by 3	(false)
b	P:	10 is divisible by 2 or 10 is divisible by 7	(true)
	not P:	10 is not divisible by 2 and 10 is not divisible by 7	(false)

Proof by contrapositive

Consider this statement:

Statement	If	it is the end of term	then	the students are happy.

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

Contrapositive	If	the students are *not* happy	then	it is *not* the end of term.

Note that the original statement and its contrapositive are logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.

- The **contrapositive** of $P \Rightarrow Q$ is the statement $(\text{not } Q) \Rightarrow (\text{not } P)$.
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

Example 9

Let $n \in \mathbb{Z}$ and consider this statement: If n^2 is even,then n is even.

a Write down the contrapositive.

b Prove the contrapositive.

Solution

a If n is odd, then n^2 is odd.

b Assume that n is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$. Squaring n gives

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 2m \in \mathbb{Z} \end{aligned}$$

Therefore n^2 is odd.

Note:

Although we proved the contrapositive, remember that we have actually proved that if n^2 is even, then n is even.

Example 10

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then n is odd.

a Write down the contrapositive.

b Prove the contrapositive.

Example 11

Let x and y be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

a Write down the contrapositive.

b Prove the contrapositive.

- Example 10

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then n is odd.

a Write down the contrapositive.

b Prove the contrapositive.

Solution

a If n is even, then $n^2 + 4n + 1$ is odd.

b Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 4n + 1 &= (2m)^2 + 4(2m) + 1 \\ &= 4m^2 + 8m + 1 \\ &= 2(2m^2 + 4m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 4m \in \mathbb{Z} \end{aligned}$$

Hence $n^2 + 4n + 1$ is odd.

Example 11

Let x and y be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

a Write down the contrapositive.

b Prove the contrapositive.

Solution

a If $\sqrt{x} \geq \sqrt{y}$, then $x \geq y$.

b Assume that $\sqrt{x} \geq \sqrt{y}$. Then $x \geq y$ by Example 3, since $\sqrt{x}$ and $\sqrt{y}$ are positive.

Section summary

- To **negate** a statement we write its opposite.
- For a statement $P \Rightarrow Q$, the **contrapositive** is the statement (not Q) $\Rightarrow$ (not P). That is, we switch the hypothesis and the conclusion and negate both.
- A statement and its contrapositive are logically equivalent.
- Proving the contrapositive of a statement may be easier than giving a direct proof.

Assigned Task

- **Cambridge Specialist Maths:**

- Exercise 6B (8 Qs)